problem considered by Desloge is not the same one discussed by Kane and Levinson. However, if one applies Kane's method to Desloge's problem, namely the one involving *arbitrary* kinematical constraints, one finds that Kane's method leads to the same equations as those obtained by Desloge using the Gibbs-Appell method, but does so with considerably less labor. Indeed, for *any* problem, Kane's method entails less labor than the Gibbs-Appell method, and the more complex the problem, the greater the savings in labor. Formulating generalized inertia forces by constructing a Gibbs function, a quantity that is of no interest in its own right, and subsequently taking partial derivatives of this function, requires substantially more labor than the simple operations one performs using Kane's method, which are the following: identify partial angular velocities and partial velocities by inspection of angular velocity and velocity expressions, and dot-multiply with inertia torques and accelerations.

Desloge also claims in Ref. 1 that use of Appell's force function to obtain generalized active forces requires less labor than Kane's method. A simple examination of the steps involved in both procedures reveals that the opposite is the case. To obtain generalized active forces with Kane's method, all one has to do is dot-multiply forces with partial velocities and torques with partial angular velocities. In constrast, use of Appell's force function requires one to first dot-multiply forces with accelerations and torques with angular accelerations, tasks significantly more laborious than the dot-multiplications performed in connection with Kane's method. This is so because partial velocities and partial angular velocities are given by simpler expressions than are accelerations and angular accelerations, and, in the case of complex dynamical systems, this disparity is enormous. But the Gibbs-Appell method requires still more work. After the lengthy dot-products have been formed to construct Appell's force function, a quantity that, like the aforementioned Gibbs function, is totally useless in its own right, one must take partial derivatives of this function. Thus, it is clear that the Gibbs-Appell method requires significantly more labor also in this regard than does Kane's method.

The use of Appell's force function suffers from yet another major shortcoming: it can lead to incorrect equations of motion when forces and torques of interest depend functionally on acceleration and/or angular acceleration, as is the case, for example, when Coulomb friction comes into play or certain feedback control laws are employed. For instance, Appell's force function must give rise to wrong equations of motion for a spinning top sliding on a rough horizontal support. Conversely, for whatever forces one encounters, Kane's method always leads to correct equations of motion. Therefore, the Gibbs-Appell method is a special case of Kane's method, not the other way around as Desloge contends.

References

[1]Desloge, E. A., "Efficacy of the Gibbs-Appell Method for Generating Equations of Motion for Complex Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 12, Jan.-Feb. 1989, pp. 114–116.

[2]Desloge, E. A., "A Comparison of Kane's Equations of Motion and the Gibbs-Appell Equations of Motion," *American Journal of Physics,* Vol. 54, May 1986, pp. 470–472.

[3]Desloge, E. A., "Relationship Between Kane's Equations and the Gibbs-Appell Equations," *Journal of Guidance, Control, and Dynamics,* Vol. 10, Jan.-Feb. 1987, pp. 120–122.

[4]Desloge, E. A., "The Gibbs-Appell Equations of Motion," *American Journal of Physics,* Vol. 56, Sept. 1988, pp. 841–846.

[5]Kane, T. R., "Rebuttal to 'A Comparison of Kane's Equations of Motion and the Gibbs-Appell Equations of Motion,' " *American Journal of Physics,* Vol. 54, May 1986, p. 472.

[6]Levinson, D. A., "Comment on 'Relationship Between Kane's Equations and the Gibbs-Appell Equations,' " *Journal of Guidance, Control, and Dynamics,* Vol. 10, Nov.-Dec. 1987, p. 593.

[7]Keat, J. E., "Comment on 'Relationship Between Kane's Equations and the Gibbs-Appell Equations,' " *Journal of Guidance, Control, and Dynamics,* Vol. 10, Nov.-Dec. 1987, pp. 594–595.

[8]Rosenthal, D. E., "Comment on 'Relationship Between Kane's Equations and the Gibbs-Appell Equations,' " *Journal of Guidance, Control, and Dynamics,* Vol. 10, Nov.-Dec. 1987, pp. 595–596.

[9]Banerjee, A. K., "Comment on 'Relationship Between Kane's Equations and the Gibbs-Appell Equations,' " *Journal of Guidance, Control, and Dynamics,* Vol. 10, Nov.-Dec. 1987, pp. 596–597.

[10]Huston, R. L., "Comment on 'Relationship Between Kane's Equations and the Gibbs-Appell Equations,' " *Journal of Guidance, Control, and Dynamics,* Vol. 11, March-April 1988, p. 120.

[11]Kane, T. R., and Levinson, D. A., "Formulation of Equations of Motion for Complex Spacecraft," *Journal of Guidance and Control,* Vol. 3, March-April 1980, pp. 99–112.

Reply by Author to David A. Levinson and Arun K. Banerjee

Edward A. Desloge*
Florida State University, Tallahassee, Florida

THE preceding Comment of Levinson and Banerjee[1] distorts my views,[2-5] hence a synopsis of my position is given before their criticisms are considered.

The standard form of the Gibbs-Appell equations of motion can be written [Ref. 5, Eq. (9)]

$$\frac{\partial}{\partial \ddot{r}_j}\left(\frac{1}{2}\sum_i m_i \ddot{x}_i^2\right) = \sum_i f_i \frac{\partial \dot{x}_i}{\partial \dot{r}_j} \tag{G}$$

If the derivative on the left-hand side of Eq. (G) is carried out and use is made of the identity $\partial \ddot{x}_i/\partial \ddot{r}_j = \partial \dot{x}_i/\partial \dot{r}_j$ [Ref. 5, Eq. (5)], one obtains Kane's equations of motion

$$\sum_i m_i \ddot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{r}_j} = \sum_i f_i \frac{\partial \dot{x}_i}{\partial \dot{r}_j} \tag{K}$$

Equations (G) and (K) are essentially the same basic equation and when applied to the same system lead to the same final result X.

For a given system there are many routes from the general equations of motion (G) to the particular equations of motion X. The force term may be evaluated directly as given, or with the help of the concept of virtual work, or with the use of generalized potentials. The differentiation on the left-hand side of Eq. (G) may be carried out immediately or after the summation has first been simplified or reformulated in any of a variety of ways. Any of these routes from Eq. (G) to X is an application of the Gibbs-Appell method. Advocates of Kane's method are not as liberal. There is an orthodox route from Eq. (K) to X, which they identify as Kane's method. The concept of virtual work is considered to be objectionable[6]; generalized potentials are considered to be useless[1]; and no thought is ever given to reformulating the inertial term to take advantage of the options available in the Gibbs-Appell form of this term.

Since the route from Eq. (G) to Eq. (K) to X is a viable option in the Gibbs-Appell method, it follows that advocates of Kane's method can never claim that their method saves more than the trivial amount of labor required to go from Eq. (G) to Eq. (K). Furthermore, to prove that the labor involved in going from Eq. (K) to X by the orthodox route is less than the labor involved in going from Eq. (G) to X by any

Received May 9, 1989; revision received July 21, 1989.

*Professor of Physics.

route would require the impossible task of considering every route from Eq. (G) to X. Hence the statement by Levinson and Banerjee that, "for any problem Kane's method involves less labor than the Gibbs-Appell method" is an unproved conjecture.

The claim made in my Note[2] is that there exists one route from Eq. (G) to X for the test system of Kane and Levinson[7] that involves far less labor than the orthodox route from Eq. (K) to X used by them, and thus their conclusion concerning the superiority of Kane's method over the Gibbs-Appell method is unjustified.

The argument of Levinson and Banerjee that my claim is without substance because I first considered a slightly more general system and then specialized to the particular system of Kane and Levinson whereas Kane and Levinson did not follow this procedure and thus did not work the same problem is farfetched. The simple fact is that I went from Eq. (G) to X with considerably less labor than Kane and Levinson went from Eq. (K) to the exact same X.

The statement by Levinson and Banerjee that I have not substantiated my claim is not true. Equations (7-16) in Ref. 2, except for two easily justified terms, can be written down immediately, using either self-evident or off-the-shelf results only, and the derivatives required to go from Eqs. (7-16) to Eqs. (17-19) in Ref. 2 can be carried out in one's head. A diagram (Fig. 1) and the simple steps sketched in the last pararaph of Sec. V in Ref. 2 allow one to pass from these equations to the particular equations of Kane and Levinson [Eqs. (63-70) in Ref. 7]. Note that there is a typographical error in the last paragraph of Sec. V in Ref. 2: the quantities x_1, x_2, x_3, r_a, and r_b in the final six expressions at the very end of the paragraph should be topped by double dots. Note also that the force expressions can be easily obtained using the relations $\sigma_1\delta x_1 + \sigma_2\delta x_2 \equiv -\sigma\delta r + 2\tau\delta\theta$, and it is not necessary to evaluate $\ddot{x}_i$ to obtain $\partial\ddot{x}_i/\partial\ddot{r}_\alpha$ since $\partial\ddot{x}_i/\partial\ddot{r}_j \equiv \partial\dot{x}_i/\partial\dot{r}_j$.) Comparison of my solution, including the last-mentioned stage, with the corresponding long, convoluted, and tedious solution in Ref. 7 using Kane's method, clearly substantiates my claim.

The aversion of Levinson and Banerjee to the use of potentials is hard for me to comprehend considering the central role played by potentials in practically every branch of physics. Their repeated warnings concerning the amount of labor required to construct the Gibbs-Appell potentials puzzle me. There is no hidden labor involved in writing down Eq. (7) in my Note[2] other than looking up a theorem in a text; and there is an insignificant amount of hidden labor involved in writing down Eq. (8) in my Note.[2]

The implication of Levinson and Banerjee that the nonapplicability of Appell's force function for acceleration-dependent forces invalidates my results is without merit. The forces in question are not acceleration dependent. Moreover, the legitimacy of acceleration-dependent forces is debatable.[8]

Throughout their Comment, Levinson and Banerjee base their judgment of the Gibbs-Appell method on an oversimplified view of the method and, as a consequence, make unwarranted assumptions concerning the difficulty of the method.

Levinson and Banerjee claim that my views have previously been *refuted* and list six cognate references[6,9-13] to back their claim. An examination of these references reveals that their claim is unjustified and in actuality my views have simply been ineffectively *disputed*. One of the six critics[10] agrees with my results but consides the difference between Eq. (G) and Eq. (K) of more significance than I do; one[6] acknowledges the basic congruence between Eqs. (G) and (K) but argues that Eq. (K) is better than Eq. (G) for two reasons, one false and the other conjectural. Two critics[11,12] overlook the identity $\partial\ddot{x}_i/\partial\ddot{r}_j \equiv \partial\dot{x}_i/\partial\dot{r}_j$, the use of which can be shown to negate their thesis that Eq. (K) is superior to Eq. (G) because it is easier to linearize than Eq. (G). The remaining two critics[9,13] sidestep the main issues, misrepresent my position, and make various unsubstantitated claims.

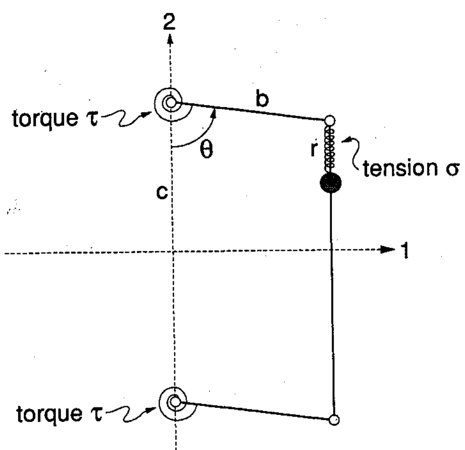

Fig. 1 The linkage involved in the specialization of Eqs. (17-19) in Ref. 2 to Eqs. (63-70) in Ref. 7.

References

[1]Levinson, D. A. and Banerjee, A. K., "Comment on 'Efficacy of the Gibbs-Appell Method for Generating Equations of Motion for Complex Systems,'" *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 2, 1990, pp. 381-382.

[2]Desloge, E. A., "Efficacy of the Gibbs-Appell Method for Generating Equations of Motion for Complex Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 1, 1989, pp. 114-116.

[3]Desloge, E. A., "Relationship Between Kane's Equations and the Gibbs-Appell Equations," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 1, 1987, pp. 120-122.

[4]Desloge, E. A., "A Comparison of Kane's Equations of Motion and the Gibbs-Appell Equations of Motion," *American Journal of Physics*, Vol. 54, No. 5, 1986, pp. 470-472.

[5]Desloge, E. A., "The Gibbs-Appell Equations of Motion," *American Journal of Physics*, Vol. 56, No. 9, 1988, pp. 841-846.

[6]Kane, T. R., "Rebuttal to 'A Comparison of Kane's Equations of Motion and the Gibbs-Appell Equations of Motion,'" *American Journal of Physics*, Vol. 54, No. 5, 1986, p. 472.

[7]Kane, T. R. and Levinson, D. A., "Formulation of Equations of Motion for Complex Spaceraft," *Journal of Guidance, Control, and Dynamics*, Vol. 3, 1980, pp. 99-112.

[8]Pars, L. A., *A Treatise on Analytical Dynamics*, Wiley, New York, 1965, Sec. 1.4.

[9]Levinson, D. A., "Comment on 'Relationship Between Kane's Equations and the Gibbs-Appell Equations,'" *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 6, 1987, p. 593.

[10]Keat, J. E., "Comment on 'Relationship Between Kane's Equations and the Gibbs-Appell Equations,'" *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 6, 1987, pp. 594-595.

[11]Rosenthal, D. E., "Comment on 'Relationship Between Kane's Equations and the Gibbs-Appell Equations,'" *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 6, 1987, pp. 595-596.

[12]Banerjee, A. K., "Comment on 'Relationship Between Kane's Equations and the Gibbs-Appell Equations,'" *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 6, 1987, pp. 596-597.

[13]Houston, R. L., "Comment on 'Relationship Between Kane's Equations and the Gibbs-Appell Equations,'" *Journal of Guidance, Control, and Dynamics*, Vol. 11, No. 2, 1988, p. 120.